LIMIT: FIRST-TIME COLLEGE PACK: ALL SLIDES AS TEXT

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1. Motivation and general idea of limit

Overview.

- In mathematics, we use the term "limit" in the sense of approaching or coming close to. We do *not* use the term limit for its other English meaning, namely, limit as a boundary or perimeter or cap.
- The notation $\lim_{x \to c} f(x) = L$ says that "the limit as x approaches c of $f(x)$ is L."
- For nice enough functions, the limit can be interpreted graphically using a "two finger test."
- The limit at a point depends only on the behavior *arbitrarily close* to the point.
- For a function to have a limit, it should be *trapped* near the point. It's not enough for it to keep coming close, then going far away.

Checkpoint questions.

- To figure out the limit of a function at 2, does the value of the function at 2.1 matter? Does the value of the function at 2.01 matter? 2.001? How close is close enough?
- What is the limit $\lim_{x \to 0} \sin(1/x)$? What's the intuitive idea behind the reasoning? More formal versions of this reasoning will be introduced after we have seen the $\varepsilon - \delta$ definition.

2. Epsilon delta definition

Overview.

- To make sense of the statement "$\lim_{x \to c} f(x) = L$," f must be a function defined "around" the point c (immediate left and immediate right) except possibly *at* c. In symbols, there exists $t > 0$ such that $(c - t, c + t) \setminus \{c\} \subseteq \operatorname{dom} f$.
- We say "$\lim_{x \to c} f(x) = L$" if for every $\varepsilon > 0$ (ε pronounced epsilon), there exists $\delta > 0$ (δ pronounced delta) such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.
- There is a uniqueness theorem for limits: if "$\lim_{x \to c} f(x) = L$" and "$\lim_{x \to c} f(x) = M$" are both true, then $L = M$. Thus, we can talk of *the* limit $\lim_{x \to c} f(x)$, which, if it exists, is unique.

Checkpoint questions.

- In order to make sense of "$\lim_{x \to c} f(x) = L$," where must the function f be defined? Must f be defined *at* c? If $f(c)$ exists, what can we say about its value?
- What's the formal definition of limit, i.e., what does "$\lim_{x \to c} f(x) = L$" mean?
- How would you write the formal definition of limit using intervals rather than absolute value inequalities to describe where x and $f(x)$ should be?
- Why is there a "$0 <$" in the inequality "$0 < |x - c| < \delta$" in the $\varepsilon - \delta$ definition? Why doesn't a "$0 <$" appear in the "$|f(x) - L| < \varepsilon$" part of the definition?
- In order to be able to talk of *the* limit $\lim_{x \to c} f(x)$, what additional fact do we need beyond the definition of what "$\lim_{x \to c} f(x) = L$" means?

3. One-sided limit definitions

Overview.

- We will make sense of "$\lim_{x \to c^-} f(x) = L$" (left hand limit) and "$\lim_{x \to c^+} f(x) = L$" (right hand limit). The definitions are similar to the two-sided limit, but the x-values being tested are restricted to the immediate left (for the left hand limit) or immediate right (for the right hand limit).
- The "left" and "right" refer to the direction of approach in the *domain*. This may or may not be the direction of approach for the function values. Whether the function value approaches its limit from the left or right depends also on the nature of the function (increasing, decreasing, etc.).

Checkpoint questions.

- In order to make sense of "$\lim_{x \to c^-} f(x) = L$," where must the function f be defined? Must f be defined *at* c? If $f(c)$ exists, what can we say about its value?
- The definitions of left hand limit, right hand limit and ordinary (two-sided) limit are pretty similar. There is only one clause that differs across the three definitions. What clause is this, and how does it differ across the definitions? Explain both in inequality notation and in interval notation.
- Why should we be careful when dealing with one-sided limits in the context of function compositions?

4. Limit game

Overview.

- We will understand the statement "$\lim_{x \to c} f(x) = L$" in terms of a game between two players, a *prover* and a *skeptic*. The prover's goal is to show the statement to be true. The skeptic's goal is to make the best possible case to show the statement to be false, by challenging the prover.
- The skeptic picks $\varepsilon > 0$, the prover picks $\delta > 0$, then the skeptic picks x such that $0 < |x - c| < \delta$. The prover wins if $|f(x) - L| < \varepsilon$. Otherwise, the skeptic wins.

- The statement "$\lim_{x \to c} f(x) = L$" is deemed true if the prover has a winning strategy for the game described above.

Checkpoint questions.

- Comparing the limit game with the formal definition of limit, we see that those choices that the skeptic makes (namely, the choice of ε and then the choice of x) are quantified by "for every" whereas those choices that the prover makes are quantified by "there exists" – why is this the case?
- Is the statement "$\lim_{x \to c} f(x) = L$" equivalent to saying that the prover wins the game? Why or why not?

5. Limit game – skeptic's victory

Overview.

- We will understand what it means for the statement "$\lim_{x \to c} f(x) = L$" to be *false* for a function f defined on $(c - t, c + t) \setminus \{c\}$ for some $t > 0$.
- We'll use the limit game: The skeptic picks $\varepsilon > 0$, the prover picks $\delta > 0$, then the skeptic picks x such that $0 < |x - c| < \delta$. The prover wins if $|f(x) - L| < \varepsilon$. This time, however, we're rooting for the skeptic to win.
- The statement we'll get is similar to the original statement, with key differences: the "for every" and "there exist" quantifiers are interchanged, and the final condition becomes $|f(x) - L| \geq \varepsilon$.

Overlooked subtlety. There are two ways in which the statement "$\lim_{x \to c} f(x) = L$" could be false:

(1) f is not defined on the immediate left or immediate right of c, i.e., there is no $t > 0$ for which f is defined on $(c - t, c + t) \setminus \{c\}$. In this case, it does not make sense to *consider* taking the limit.
(2) f is defined on the immediate left and right of c, but the statement is still false.

The video focuses on (2). While shooting the video, I forgot to clarify the scope or explicitly exclude case (1).

Checkpoint questions.

- Comparing the definitions for "$\lim_{x \to c} f(x) = L$" to be true and false, why do the quantifiers "for every" and "there exists" get interchanged?
- Assume that f is defined on the immediate left and right of c. Is the statement "$\lim_{x \to c} f(x) = L$" being false equivalent to saying that the skeptic wins the game? Why or why not?

6. Non-existence of limit

Overview.

- We will understand what it means for a function f defined on $(c - t, c + t) \setminus \{c\}$ for some $t > 0$ to *not* have a limit at c.
- We will look at the example $f(x) := \sin(1/x)$ and $c = 0$.
- We'll use the limit game: The skeptic picks $\varepsilon > 0$, the prover picks $\delta > 0$, then the skeptic picks x such that $0 < |x - c| < \delta$. The prover wins if $|f(x) - L| < \varepsilon$.

- Based on the nature of the function, we will construct an explicit winning strategy for the skeptic.

Scope clarification. There are two ways in which $\lim_{x \to c} f(x)$ may not exist.

(1) f is not defined on the immediate left or immediate right of c, i.e., there is no $t > 0$ for which f is defined on $(c - t, c + t) \setminus \{c\}$. In this case, it does not make sense to *consider* taking the limit.
(2) f is defined on the immediate left and right of c, but the limit still does not exist.

The example in this video illustrates (2).

Checkpoint questions. Just try repeating the reasoning used in the video (without looking at the video) to justify why $\lim_{x \to 0} \sin(1/x)$ does not exist.